Math Readiness Problems for the AI program

Learners entering the AI executive education program should have an appropriate mathematical background in multivariable calculus, linear algebra, probability, and statistics. Such learners should be able to solve most of the following problems. It may be necessary for some to first review some of their undergraduate material. In particular, this may be the case for prospective learners who have been working in industry and have not been in academemia for some time.

1. Solve for the scalar values μ and σ^2, in terms of x_i and n, which maximize the following expression:

$$\log \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \right)$$

2. We are given the equations of two hyperplanes:

$$\mathbf{w} \cdot \mathbf{x} - b = 1$$
$$\mathbf{w} \cdot \mathbf{x} - b = -1$$

   b is a scalar, $\mathbf{w}$ is a constant n-vector, and $\mathbf{x}$ is an n-vector $(x_1, ..., x_n)^T$. Find the distance between the two hyperplanes in terms of $\mathbf{w}$ and b.

3. Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$, where $m > n$, and a vector $\mathbf{b} \in \mathbb{R}^m$. Describe how to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that the distance between $A\mathbf{x}$ and $\mathbf{b}$ is minimized.

4. Suppose we have a matrix $A \in \mathbb{R}^{n \times n}$, in which all columns sum to 1 and all matrix elements are non-negative. Describe how to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x}$ (assuming it exists).

5. Two random variables X, Y follow a joint probability density function

$$f(x, y) = \begin{cases} c(4x^2y + y^2), & x \in [0, 1], y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

   (a) Compute $\Pr(X + Y > 1.5)$.
   (b) Compute $E[X]$ and $E[Y]$.
   (c) Find $Cov(X, Y)$.
   (d) Find $f(x|y)$, the conditional pdf of x given y.
   (e) Compute $E[X \mid Y = 0.5]$.